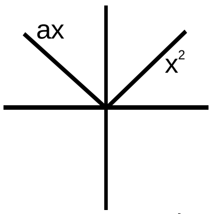
EXERCISE – III HINTS & SOLUTIONS

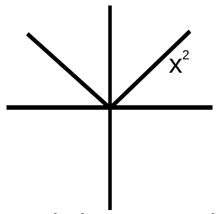
Sol.1 $f(x) = \begin{cases} x^2 & x \ge 0 \\ ax & x < 0 \end{cases}$

If a < 0



Non-Monotonic

If a > 0



Strictly monotonically Increasing at x= 0

$\boxed{a \in R^+}$

Sol.2 **(i)** $f(x) = \tan^{-1}(\sin x + \cos x)$

$$f'(x) = \frac{\cos x - \sin x}{1 + (\sin x + \cos x)^2}$$

For f(x) is a decreasing function

$f'(x) < 0$

$\cos x - \sin x < 0$

$$\frac{1}{\sqrt{2}} \cos x - \frac{1}{\sqrt{2}} \sin x < 0$$

$$\cos\left(x + \frac{\pi}{4}\right) < 0$$

$$\frac{\pi}{2} < x + \frac{\pi}{4} < \frac{3\pi}{2}$$

$$\frac{\pi}{2} - \frac{\pi}{4} < x < \frac{3\pi}{4} - \frac{\pi}{4}$$

$$\frac{\pi}{4} < x < \frac{5\pi}{4}$$

(ii) $f(x) = \frac{5\pi}{4} - \ln(1 + x)$ D_f $\boxed{x > -1}$

$$f'(x) = \frac{\sqrt{1+x} - \frac{x}{2\sqrt{1+x}}}{(1+x)} - \frac{1}{1+x}$$

$$= \frac{2(1+x) - x}{2(1+x)^{3/2}} - \frac{1}{1+x}$$

$$= \frac{2 + x - 2\sqrt{1+x}}{2(1+x)^{3/2}}$$

$$= \frac{(\sqrt{1+x} - 1)^2}{2(1+x)^{3/2}} > 0 \text{ for } x > -1$$

f(x) is increasing function for > –1

Sol.3 $f(x) = x^3 + (a - 1)x^2 + 2x + 1$

$f'(x) = 3x^2 + 2(a - 1)x + 2 > 0$

$D \le 0$

$4(a - 1)^2 - 4(2)(3) \le 0$

$(a - 1)^2 - 6 \le 0$

$-\sqrt{6} \le a - 1 \le \sqrt{6}$

$1 - \sqrt{6} \le a \le 1 + \sqrt{6}$

Sol.4 **(i)** $f(x) = \frac{x^4}{4} + \frac{x^3}{3} - 3x^2 + 5$

$f'(x) = x^3 + x^2 - 6x$

$= x(x^2 + x - 6)$

$= x(x + 3)(x - 2)$

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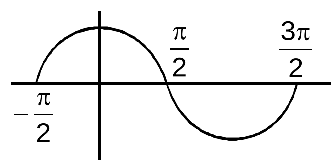
MI $x \in (-3, 0) \cup (2, \infty)$

MD $x \in (-\infty, -3) \cup (0, 2)$

(ii) $f(x) = \sin \frac{\pi}{x}$

$$f'(x) = -\frac{\pi}{x^2} \cos \frac{\pi}{x}$$



Function will be decreasing for $\cos \frac{\pi}{x} \ge 0$

$$-\frac{\pi}{2} < \frac{\pi}{x} < \frac{\pi}{2}$$

$$2n\pi - \frac{\pi}{2} < \frac{\pi}{x} < 2n\pi + \frac{\pi}{2}$$

$$\frac{4n-1}{2} < \frac{1}{x} < \frac{4n+1}{2}$$

$$\boxed{\frac{2}{4n+1} < x < \frac{2}{4x-1}} \qquad n \in I$$

function will be increasing if $\cos\frac{\pi}{x} < 0$

$$\frac{\pi}{2} < \frac{\pi}{x} < \frac{3\pi}{2}$$

$$2n\pi + \frac{\pi}{2} < \frac{\pi}{x} < 2n\pi + \frac{3\pi}{2}$$

$$\frac{4n+1}{2} < \frac{1}{x} < \frac{4n+3}{2} \qquad n \in I$$

$$\boxed{\frac{2}{4n+3} < x < \frac{2}{4n+1}} \qquad \textbf{Ans.}$$

(iii) $f(x) = \log_3^2 x + \log_3 x \qquad D_f : \boxed{x > 0}$

$$f(x) = \left(\frac{\ell nx}{\ell n3}\right)^2 + \frac{\ell nx}{\ell n3}$$

$$2\left(\frac{\ln x}{\ln 3}\right).\frac{1}{x} + \frac{1}{x.\ln 3} = 0$$

$$\ln x = \frac{-1}{2}\ln 3$$

$$x = \frac{1}{\sqrt{3}}$$

M.D. $\quad x \in \left(0, \frac{1}{\sqrt{3}}\right)$

M.I. $\quad x \in \left(\frac{1}{\sqrt{3}}, \infty\right)$

Sol.5 $f(x) = (a + 2)x^3 - 3ax^2 + 9ax - 1$

C – 1 When $a = -2$

$f(x) = 6x^2 - 18x - 1$

When can't be monotonic for

C–2 $a \neq -2$

$f'(x) < 0$

$3(a + 2)x^2 - 6ax + 9a < 0 \ \forall x \in R$

$\boxed{a < -2}$ & $D < 0 \ b^2 - 4ac$

$36a^2 - 36.3a(a + 2) < 0$

$a^2 - 3a^2 - 6a < 0$

$2a(a + 3) > 0$

$a > 0$ or $a < -3$

$a \in (-\infty, -3)$ check for end value

$a = -3 \ f'(x) = -3x^2 + 18x - 27 = -(x-3)^2 = 0$

at $x = 3$

$a \in (-\infty, -3)$

Sol.6 $f(x) = \sin^{-1}\frac{x}{\sqrt{x^2+1}} - \ell nx \qquad D_f : x > 0$

$f(x) = \tan^{-1} x - \ell nx$

$$f'(x) = \frac{1}{1+x^2} - \frac{1}{x}$$

$$= \frac{x-1-x^2}{x(1+x^2)} = -\left(\frac{x^2-x+1}{x(1+x^2)}\right)$$

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So f(x) is decreasing in its domain

$$\text{Greatest value } f\left(\frac{1}{\sqrt{3}}\right) = \tan^{-1}\frac{1}{\sqrt{3}} - \ell n\frac{1}{\sqrt{3}}$$

$$= \tan^{-1}\sqrt{3} - \ell n\sqrt{3}$$

$$= \frac{\pi}{3} - \frac{1}{2}\ell n3 \qquad \textbf{Ans.}$$

Sol.7 let $h(x) = gof(x)$

$h'(x) = g'(f(x)).\ f'(x) < 0$

$> 0 \quad < 0$

h is decreasing function.

Sol.8 **(i)** P.T. $x < -\ell n(1 - x) < \frac{x}{1-x}$ For $0 < x < 1$

$f(x) = x + \ell n(1 - x)$

$$f'(x) = 1 - \frac{1}{1-x} = \frac{-x}{1-x} < 0$$

For $0 < x < 1$

$f(x) < f(0)$

$x + \ell n(1 - x) < 0$

$x < -\ell n(1 - x)$ H.P.

$$g(x) = \frac{x}{1-x} + \ell n(1-x)$$

$g(x)\uparrow$

$g(x) > g(0)$

$g(x) > 0$

$$\frac{x}{1-x} > -\ell n(1-x)$$

Sol.9 $\frac{\tan x_2}{\tan x_1} > \frac{x_2}{x_1}$ for $0 < x_1 < x_2 < \frac{\pi}{2}$

Let $f(x) = \frac{\tan x}{x}$

$$f'(x) = \frac{x \sec^2 x - \tan x}{x^2}$$

$$= \frac{x - \sin x \cos x}{(x \cos x)^2} = \frac{x - \frac{\sin 2x}{2}}{(x \cos x)^2}$$

Let $g(x) = x - \frac{\sin 2x}{2}$

$g'(x) = 1 - \cos 2x$

$= 2 \sin^2\pi > 0$

$g(x)\uparrow$

$g(x) > g(0)$

$\Rightarrow$ $g(x) > 0$

$f'(a) > 0$

$f(x_2) > f(x_1) \Rightarrow \frac{\tan x_2}{x_2} > \frac{\tan x_1}{x_1}$

$\frac{\tan x_2}{\tan x_1} > \frac{x_2}{x_1}$ H.P.

Sol.10 Let $f(x) = 2 \sin x + \tan x - 3x$

$f'(x) = 2\cos x + \sec^2 x - 3$

$$= \frac{(2\cos x + 1)(\cos x - 1)^2}{\cos^2 \pi}$$

$f'(x) > 0$

$f(x) > f(0)$

$2 \sin x + \tan x > 3x$ H.P.

$$\frac{3x}{2\sin x + \tan x} < 1$$

$$\lim_{x \to 0} \left[\frac{3x}{2\sin x + \sin x}\right] = 0$$

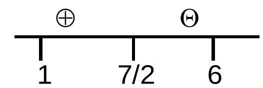
Sol.11 $f(x) = \sqrt{x-1} + \sqrt{6-x}$; $x - 1 \geq 0$; $6 - x \geq 0$

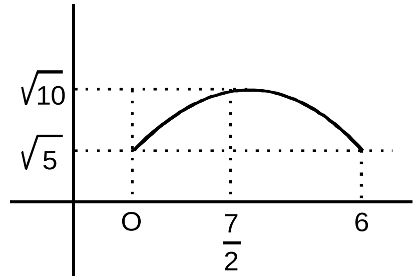
$x \geq 1$ $x \leq 6$

of x t [1, 6]

$$f'(x) = \frac{1}{2\sqrt{x-1}} - \frac{1}{2\sqrt{6-x}} = 0$$

$\Rightarrow x = \frac{7}{2}$





$f(1) = \sqrt{5}$

$f(6) = \sqrt{5}$

$f\left(\frac{7}{2}\right) = \sqrt{10}$

Range is $[\sqrt{5}, \sqrt{10}]$

Sol.12 $f(x) = \frac{e^{x^2} - e^{-x^2}}{e^{x^2} + e^{-x^2}}$

P.T. Injectives $\rightarrow$ One-One $\rightarrow$ Monotonic

$f(t) = y = \frac{t-1}{t+1}$ $t = e^{2x^2}$

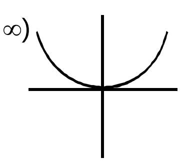
$f'(t) = \frac{2}{(t+1)} > 0$ $g(x) = e^{2x^2}$

$t\uparrow f(t)\uparrow$ $g'(x) = 4x\, e^{2x^2}$

But For $[0, \infty)$

It is $\uparrow$

as $x\uparrow$ $t\uparrow$



f(x) is one-one

Sol.13 P.T. $e^x + \sqrt{1+e^{2x}} \geq (1 + x) + \sqrt{2 + 2x + x^2}$ $\forall x \in R$

$e^x + \sqrt{1+e^{2x}} \geq (1 + x) + \sqrt{1+(1+x)^2}$

Let $f(t) = t + \sqrt{1+t^2}$ We wish to prove that $f(e^x) \geq f(x + 1)$

$$f'(t) = 1 + \frac{2t}{2\sqrt{t^2+1}} = \frac{t + \sqrt{t^2+1}}{\sqrt{t^2+1}}$$

If $t > 0$, $f'(t) > 0$

If $t < 0$ Let $t = -a$ $\quad f'(t) = \dfrac{\sqrt{a^2+1}-a}{\sqrt{a^2+1}} > 0$

$f'(t) > 0$ always

f(t) is incrasing function

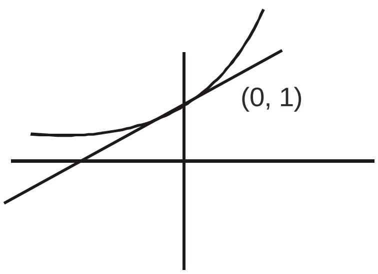
We wish to Prove that

$f(e^x) \geq f(x + 1)$

To Prove that

$e^x \geq (1 + x)$ ....(1)

Always for all $x \in R$



so $f(e^x) \geq f(1 + x)$

Sol.14 Given $f'(\sin x) < 0$ and $f''(\sin x) > 0$, & $x \in \left(0, \frac{\pi}{2}\right)$

$g(x) = f(\sin x) + f(\cos x)$

$g'(x) = \cos x\, f'(\sin x) - \sin x\, f'(\cos x)$

$g'\left(\frac{\pi}{4}\right) = 0$ (By observation)

$g''(x) = -\sin x\, f'(\sin x) + \cos^2 x\, f''(\sin x) - \cos x$

$f'(\cos x) + \sin^2 x + f''(\cos x)$

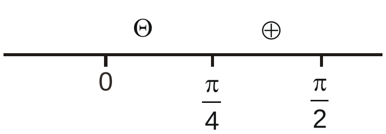
$= \cos^2 x\, f''(\sin x) + \sin^2 x\, f''(\cos x) - \underbrace{\overset{\oplus}{\sin x}\, \overset{\ominus}{f'(\sin x)}}_{\oplus}$

$\underbrace{-\overset{\oplus}{\cos x}\, \overset{\Theta}{f'(\cos x)}}_{\oplus}$

$g''(x) > 0$

$g'(x) \Uparrow g'(x) > g'(0)$

$g'(x) > 0 \Rightarrow g'(x) \uparrow$ increasing



decreasing in $\left(0, \frac{\pi}{4}\right)$

Sol.15 If $ax^2 + \dfrac{b}{x} \geq c \;\forall\; x > 0,\ a > 0,\ b > 0$

P.T. $27ab^2 \geq 4c^3$

$ax^3 + b - cx \geq 0$

$f(x) = ax^3 - cx + b$

$f'(x) = 3ax^2 - c = 0 \Rightarrow x^2 = \dfrac{c}{3a} \Rightarrow x = \pm\sqrt{\dfrac{c}{3a}}$

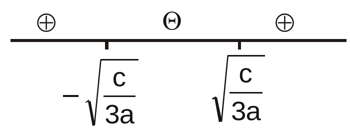
Case–I $c < 0$

$ax^3 - cx + b$ wil always be positive

Case II

$c > 0$

$x = \pm\sqrt{\dfrac{c}{3a}}$



$f\left(\sqrt{\dfrac{c}{3a}}\right) \geq 0$

$a\left(\dfrac{c}{3a}\right)^{3/2} + b - c\left(\dfrac{c}{3a}\right)^{1/2} \geq 0$

$a\left(\dfrac{c}{3a}\right)^{3/2} - c\left(\dfrac{c}{3a}\right)^{1/2} \geq -b$

Squaring both side

$a^2\left(\dfrac{c}{3a}\right)^3 + c^2\left(\dfrac{c}{3a}\right) - 2ac\left(\dfrac{c}{3a}\right)^2 \leq b^2$

$\boxed{4c^3 \leq 27ab^2}$ H.P.

Sol.16 $f(x) = \sin 2x - 8(a + 1)\sin x + (4a^2 + 8a - 14)x$

$f'(x) = 2\cos 2x - 8(a + 1)\cos x + (4a^2 + 8a - 14)$

$= 2(2\cos^2 x - 1) - 8(a + 1)\cos x + (4a^2 + 8a - 14)$

$= 4[\cos^2 x - 2(a + 1)\cos x + (a^2 + 2a - 4)]$

For monotonic $\uparrow$ and no critical points

$f'(x) > 0 \;\forall\; x \in R$

$D = 4(a + 1)^2 - 4(a^2 + 2a - 4)$

$= 4a^2 + 8a + 4 - 4a^2 - 8a + 16 = 20$

Which is always positive.

Let $\cos x = y;\ y \in [-1, 1]$

$g(y) = y^2 - 2(a+1)y + (a^2 + 2a - 4)$

We have to find out those values of a for which

g(y) >) for all $y \in (-1, 1)$

(i) $g(-1) > 0$ & $\frac{-b}{2a} < -1$

$1 + 2(a+1) + a^2 + 2a - 4 > 0$

$a^2 + 4a - 1 > 0$

$a > -2 + \sqrt{5}$ or $a < -2\ \sqrt{5}$

$\frac{-b}{2a} < -1$

$\frac{b}{2a} > 1$

$\frac{-2(a+1)}{2a} > 1$

$a + 1 < -1$

$a \in (-\infty, -2-\sqrt{5})$

(ii) $1 - 2(a+1) + a^2 + 2a - 4 > 0$ $\frac{b}{2a} < -1$

$a^2 - 5 > 0$ $\qquad -\frac{2(a+1)}{2} < -1$

$a > \sqrt{5}$

$a + 1 > 1$

$a > 0$

Final answer $a \in (-\infty, -2-\sqrt{5}) \cup (\sqrt{5}, \infty)$

Sol.17 $f(x) = \frac{ax^3}{3} + (a+2)x^2 + (a-1)x + 2$

$f'(x) = ax^2 + 2(a+2)x + (a-1)$

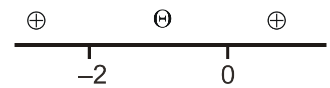
$f''(x) = 2ax + 2(a+2)$

For point of inflection

$2ax + 2(a+2) = 0$

Negative point of inflection

$-\left(\frac{a+2}{a}\right) < 0$



$\frac{a+2}{a} > 0$

$a \in (-\infty, -2) \cup (0, \infty)$

Sol.18 If we assume

$\ell n(1+x) > \frac{\tan^{-1} x}{1+x}$

$(1+x)\ \ell n(1+x) > \tan^{-1} x$

Now $f(x) = (1+x)\ \ell n(1+x) - \tan^{-1} x$

$f'(x) = \frac{1+x}{1+x} + \ell n(1+x) - \frac{1}{1+x^2}$

$= \underbrace{1+\ell n(1+x)}_{>1} - \frac{1}{\underbrace{1+x^2}_{<1}} \geq 0\ \forall\ x \in R$

f(x) is increasing

$f(x) > f(0)$

$\boxed{f(x) > 0}$ H.P.

Sol.19 P.T. $\frac{\tan x}{x} > \frac{x}{\sin x}$ for $x \in \left(0, \frac{\pi}{2}\right)$

TPT $\tan x \sin x - x^2 > 0$

Let $f(x) = \tan x \sin x - x^2$

$f'(x) = \tan x \cos x + \sec^2 x + \sin x - 2x$

$f''(x) = \cos x + \cos x \sec^2 x + 2 \sin x \sec^2 x \tan x - 2$

$= \underbrace{\left(\cos x + \frac{1}{\cos x} - 2\right)}_{\text{Positive}} + \underbrace{2 \sin x \sec^2 x \tan x}_{\text{Positive}}$

$f''(x) > 0$

$f'(x) \uparrow$ing

$f'(x) > f'(0)$

$f'(x) > 0 \Rightarrow f(x)$ increasing

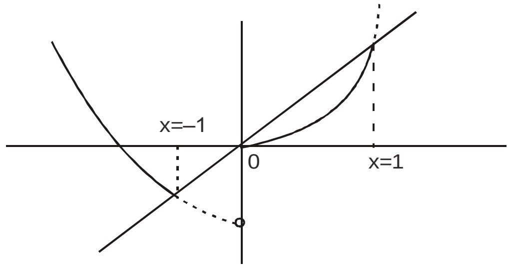
H.P.

$f(x) > f(0)$

$f(x) > 0$

$\tan x \sin x > x^2$ H.P.

Sol.20



non monotonic at x = – 1

$\uparrow$ at x = 0, 1

Sol.21 $f(x) = \sin x - a \sin 2x - \dfrac{1}{3}\sin 3x + 2ax$

$f'(x) = \cos x - 2a \cos 2x - \cos 3x + 2a \geq 0$

$= 4a \sin^2 x + 2 \sin 2x \sin x$

$= 4a \sin^2 x + 4 \sin^2 x \cos x$

$= 4 (a + \cos x) \sin^2 x$

$a \in [1, \infty)$ It won't become negative

So $f'(x) \geq 0$

Sol.22 **(i)** Let $f(x) = 1 + x^2 - x \sin x - \cos x$

$f'(x) = 2x - \sin x - x \cos x + \sin x$

$f'(x) = x (2 - \cos x) \geq 0$ for $\forall\ x \geq 0$

f(x) is increasing

$f(x) \geq f(0)$

$1 + x^2 - x \sin x - \cos x \geq 0$ H.P.

(ii) $f(x) = \sin x - \sin 2x - 2x$

$f'(x) = \cos x - 2 \cos 2x - 2$

$= \cos x - 2 (2 \cos^2 x - 1) - 2$

$= \underbrace{\underbrace{\cos x}_{>0}\underbrace{(1 - 4\cos x)}_{<0}}_{<0} < 0$

$f(x)\downarrow$

$f(x) \leq f(0)$

$\boxed{f(x) \leq 0}$ H.P.

(iii) Let $f(x) = \dfrac{x^2}{2} + 2x + 3 - (3 - x) e^x$

$f'(x) = x + 2 - (3 - x) e^x + e^x$

$f''(x) = e^x (x + 1) - 1 > 0$

$f'(x)\uparrow$

$f'(x) > f'(0)$

$f(x) \uparrow$ing

$f(x) > f(0)$

$\boxed{f(x) \geq 0}$ H.P.

Sol.23 Let $f(x) = x \sin x - \dfrac{\sin^2 x}{2}$

$f'(x) = \sin x + x \cos x - \dfrac{2 \sin x \cos x}{2}$

$f'(x) = \sin x + x \cos x - \sin x \cos x$

$= \cos x (x - \sin x) + \sin x$

Let $g(x) = x - \sin x$

$g'(x) = 1 - \cos x > 0$ for $x \in \left(0, \dfrac{\pi}{2}\right)$

$g(x) \uparrow$ing

$g(x) > g(0) \Rightarrow \boxed{g(x) > 0}$

So $f'(x) > 0$

f(x) is increasing

Least value = f(0) = 0

Greatest value $= f\left(\dfrac{\pi}{2}\right) = \dfrac{\pi}{2} - \dfrac{1}{2} = \dfrac{1}{2}(\pi - 1)$

$0 < f(x) < \dfrac{1}{2}(\pi - 1)$

Sol.24 $f(x) = f'(x) = 3\left(1 - \dfrac{\sqrt{21 - 4b} - b^2}{b + 1}\right) x^2 + 5 > 0$

$3\left(1 - \dfrac{\sqrt{21 - 4b - b^2}}{b + 1}\right) > 0$

$21 - 4b - b^2 \geq 0$

$b \in [-7, 3]$

But

for $b \in [-7, -1] \cup [2, 3]$

co-efficient of quadrate > 0

so f(x) is increasing at every point of It's domain

$b \in [-7, -1) \cup [2, 3]$

Sol.25 **(a)** Let $f(x) = x^2 - (1 + x)\,[\ell n\,(1 + x)]^2$

$f'(x) = 2x - [\ell n\,(1 + x)]^2 - 2\ell n\,(2x - 4) = 0$

$\Rightarrow \boxed{x = 2}$

Increasing in $(2, \infty)$

Decreasing is $(-\infty, 2)$

(b) $f(x) = \dfrac{e^x}{x}$

$f'(x) = \dfrac{xe^x - e^x}{x^2} = 0 \Rightarrow x = 1$

$\uparrow$ is $(1, \infty)$

$\downarrow$ is $(-\infty, 1) - \{0\}$

(c) $f(x) = x^2\,e^{-x}$

$f'(x) = 2x\,e^{-x} - x^2\,e^{-x}$

$= -\,e^{-x}\,(x\,(x-2))$

$\ominus \quad \oplus \quad \ominus$

0 2

$\uparrow$ is $(0, 2)$

$\downarrow$ is $(-\infty, 0) \cup (2, \infty)$

Sol.26

(a) $f'(x) = 2\,e^{x^2-4x}\,(2x - 4) = 0 \Rightarrow x = 2$

increasing in $(2,\infty)$

decreasing in $(-\infty, 2)$

(b) $f'(x) = \dfrac{xe^x - e^x}{x^2} = 0 \Rightarrow x = 1$

increasing in $(1,\infty)$

decreasing in $(-\infty, 1) - \{o\}$

(c) $f'(x) = 2x\,e^{-x} - x|2\ e^{-x} = e^{-x}\,x\,(2 - x)$

increasing is $(0, 2)$

decreasing in $(-\infty, 0)$ u $(2,\infty)$

(d) $f'(x) = 4x - \dfrac{1}{x} = \dfrac{4x^2 - 1}{x} = \dfrac{(2x-1)(2x\ H)}{x}$

increasing is $\left(\dfrac{-1}{2}, 0\right) \cup \left(\dfrac{1}{2}, \infty\right)$

decreasing $\left(-\infty, \dfrac{-1}{2}\right) \cup \left(0, \dfrac{1}{2}\right)$

Sol.27

$f'(x) = -1 - 3x^2 < 0 \Rightarrow$ f is an decreasing fn.

$1 - f(x) - f^3(x) > f(1 - 5x)$

$f(f(x)) > f(1 - 5x)$

$f(x) < 1 - 5x$

$1 - x - x^3 < 1 - 5x$

$x^3 - 4x > 0 \Rightarrow x\,(x + 2)\,(x - 2) > 0$

$(-2, 0) \cup (2,\infty)$ $\quad - \quad + \quad - \quad +$

$-2 \quad 0 \quad 2$

Sol.28 **(a)** $f(x) = \sin x - \cos x$ is $[0, 2\pi]$

$f'(x) = \cos x + \sin x = 0$

$x = \dfrac{3\pi}{4}\ \dfrac{7\pi}{4}$

$\oplus \quad \ominus \quad \oplus$

$0 \quad \dfrac{3\pi}{4} \quad \dfrac{7\pi}{4} \quad 2\pi$

$f''(x) = -\sin x + \cos x$

$f''\left(\dfrac{3\pi}{4}\right) = -\dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}} < 0$

$\uparrow$ is $\left[0, \dfrac{3\pi}{4}\right) \cup \left[\dfrac{\pi}{4}, 2\pi\right)$

$\downarrow$ is $\left(\dfrac{3\pi}{4}, \dfrac{7\pi}{4}\right)$

(b) $g(x) = 2\sin x + \cos 2x \quad [0, 2\pi]$

$g'(x) = 2\cos x - 2\sin 2x$

$= 2\cos x - 4\sin x \cos x$

$= 2\cos x\,(1 - 2\sin x)$

$\cos x = 0; \qquad \sin x = \dfrac{1}{2}$

$\Rightarrow x = \dfrac{\pi}{2}\ \dfrac{3\pi}{2} \qquad x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

$\oplus \quad \ominus \quad \oplus \quad \ominus \quad \oplus$

$0 \quad \dfrac{\pi}{6} \quad \dfrac{\pi}{2} \quad \dfrac{5\pi}{6} \quad \dfrac{3\pi}{2} \quad 2\pi$

$\uparrow$ing is $\left[0, \dfrac{\pi}{6}\right) \cup \left(\dfrac{\pi}{6}, \dfrac{5\pi}{6}\right) \cup \left(\dfrac{3\pi}{2}, 2\pi\right]$

$\downarrow$ing is $\left(\dfrac{\pi}{6}, \dfrac{\pi}{2}\right) \cup \left(\dfrac{5\pi}{6}, \dfrac{3\pi}{2}\right)$

Sol.29 $f(x) = x^3 - x^2 + x + 1$

$f'(x) = 3x^2 - 2x + 1$

$D = 4 - 12 < 0$

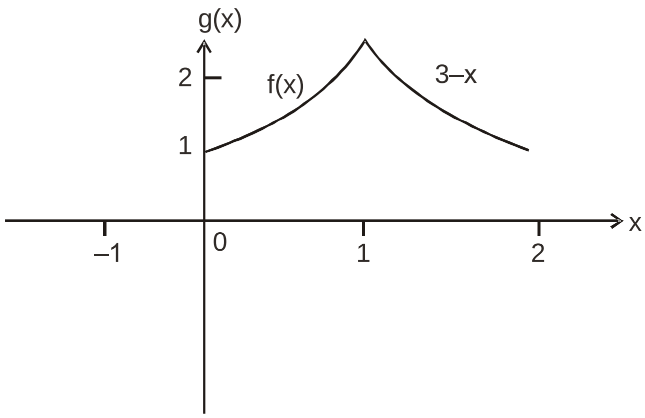
$f'(x) > 0$

f(x) willbe increasing

$g(x) = f(x) \quad 0 \le x \le 1$

$3 - x \quad 1 < x \le 2$

continous at x = 1, but not differentiable



Sol.30 **(a)** $f(x) = 12\, x^{4/3} - 6\, x^{1'3}\ x \in [-1, 1]$

$f'(x) = 12 \times \frac{4}{3} x^{1/3} - \frac{6}{3} x^{-2/3}$

$= 16\, x^{1/3} - 2\, x^{-2/3}$

$= 2\left[\frac{8x-1}{x^{2/3}}\right] = 0 \Rightarrow \boxed{x = \frac{1}{8}}$

$f(-1) = 12 + 6 = 18 \rightarrow$ Greatest value

$f(1) = 12 - 6 = 6$

$f\left(\frac{1}{8}\right) = 12\left(\frac{13}{2}\right)^{4/3} - 6\left(\left(\frac{1}{2}\right)^3\right)^{1/3}$

$= \frac{12}{16} - \frac{6}{2} = \frac{3}{4} - 3 = -\frac{9}{4} \rightarrow$ least value.

(b) $f(x) = x^5 - 5x^4 + 5x^3 + 1$

$f'(x) = 5x^4 - 2ax^3 + 15\, x^2 \quad [-1, 2]$

$f'(x) = 5x^2 (x^2 - 4x + 3)$

$= 5x^2 (x - 1)(x - 3)$

Critical points x = 0, 1, 3

$= -10$

$f(0) = 1$

$f(1) = 2$

$f(2) = -6$

Maximum at $(x = 1) = 2$

Maximum (at $x = -1) = -10$

Sol.31 $f(x) = \left(\frac{a^2-1}{3}\right) x^3 + (a - 1)\, x^2 + 2x + 1$

case–I

$a = 1$

$f(x) = 2x + 1$

which is always increasing

case–II

$a = -1$

$f(x) = -2x^2 + 2x + 1$

which is no-monotonic

Case–III

$a \ne -1$

$f'(x) = (a^2 - 1)\, x^2 + 2\,(a - 1)\, x + 2$

$D < 0$

$4\,(a - 1)^2 - 42\,(a^2 - 1) < 0$

$(a - 1)(a + 3) > 0$

$a > 1,\ a < -3$

check for $a = -3$

$f'(x) = 8x^2 - 8x + 2 = 2\,(4x^2 - 4x + 1)$

$= 2\,(2x - 1)^2 > 0$

So final answer $a \in (-\infty, -3] \cup [1, \infty)$

Sol.32 $f(x) = x^3 + (2a + 3)\, x^2 + 3\,(2a + 1)\, x + 5$

$f'(x) = 3x^2 + 2\,(2a + 3)\, x + 3\,(2a + 1)$

$D < 0$

$4(2a + 3)^2 - 4.9\,(2a + 1) < 0$

$4a^2 - 6a < 0$

$a\,(2a - 3) < 0$

$\boxed{0 < a < \frac{3}{2}}$

Check for a = 0

$f'(x) = 3x^2 + 6x + 3 = 3\,(x^2 + 2x + 1)$

$= 3\,(x + 1)^2 > 0$

Check for $a = \frac{3}{2}$

$f'(x) = 3x^2 + 12\, x + 12\ 3\,(x^2 + 4x + 4)$

$= 3(x+2)^2 > 0$

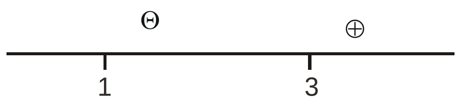
So final answer $a \in \left[0, \frac{3}{2}\right]$

Sol.33 $f(x) = \int_x^{x^2} \frac{1}{t}\ell n\left(\frac{t-1}{32}\right)dt$ $\quad D_f$ $\boxed{x > 1}$

$f'(x) = \frac{1}{x^2}\left[\ell n\left[\left(\frac{x^2-1}{32}\right)^2 \times \frac{32}{(x-1)}\right]\right] = 0$

$\frac{(x-1)(x+1)^2}{32} = 1$

$(x-1)(x+1)^2 = 32 \Rightarrow x = 3$



$\uparrow$ is $(3, \infty)$

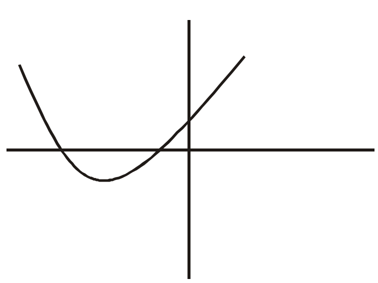
$\downarrow$ is $(1, 3)$

Sol.34 $f(x) = 2e^x - ae^{-x} + (2a+1)x - 3$

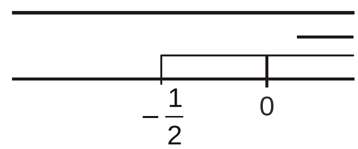
$f'(x) = 2e^x + ae^{-x} + (2a+1)$

$= e^{-}(2e^{2x}(2a+1)e^x + a)$

$D < 0$	$e^x = t \Rightarrow t \in (0, \infty)$
$(2a+1)^2 - 8a < 0$	$g(t) = 2t^2 + (2a+1)t + a$
$(2a-1)^2 < 0$	$g(0) > 0 \Rightarrow a > 0$
$a \in \phi$	$D > 0 \Rightarrow (2a-1)^2 > 0$
	$\Rightarrow \boxed{a \in R}$
No Solution	$\frac{-b}{2a} < 0 \Rightarrow -\left(\frac{2a+1}{4}\right) < 0$
	$a + \frac{1}{2} > 0$



$a > -\frac{1}{2}$



$\boxed{a > 0}$

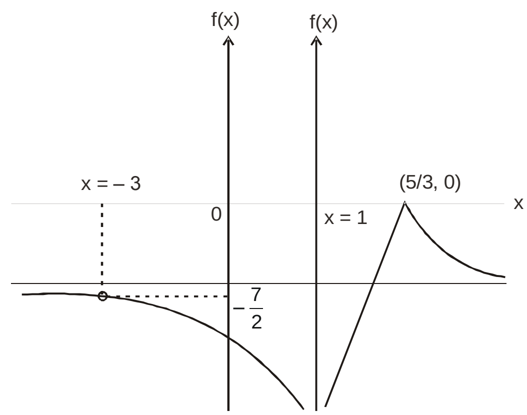
check for $a = 0$

$g(t) = 2t^2 + t$

$\boxed{D < 0}$

Final answer $\boxed{a \geq 0}$

Sol.35 $f(x) = -\left|\frac{x^2-9}{x+3} - x + \frac{2}{x-1}\right|$



(a) Range $y \in (-\infty, 0]$

(b) $\uparrow$ is $\left(1, \frac{5}{3}\right)$ and $\downarrow$ is $(-\infty, 1) \cup \left(\frac{5}{3}, \infty\right) - \{-3\}$

(c) $x = \frac{5}{3}$

(d) Removable discont. at $x = -3$

(e) $y = -\left(\frac{x - \frac{5}{3}}{x-1}\right)$

$y' = -3\left(\frac{-1+\frac{5}{3}}{(x-1)^2}\right) = \left.\frac{-2}{(x-1)^2}\right|_{x=0} = -2$

Sol.36 Let $f(x) = x^2 - 1 - 2x\,\ell nx$

$f'(x) = 2x - 2\,\ell n\,x - 2$

$$f''(x) = 2 - \frac{2}{x} = 2\left(\frac{x-1}{x}\right) > 0 \text{ for } x > 1$$

$f'(x)\uparrow$

$f'(x) > f'(1)$

$f'(x) > 0 \Rightarrow f(x)$ is $\uparrow$ing

$f(x) > f(1)$

$f(x) > 0$ H.P.

Let $g(x) = 2x\,\ell nx - 4x\,(x-1) + 2\,\ell nx$

$$g'(x) = 2\ell nx - 2 - 4\,(2x-1) + \frac{2}{x}$$

$$g''(x) = \frac{2}{x} - 8 - \frac{2}{x^2}$$

$$= \frac{-8x^2 + 2x - 2}{x^2} \text{ always +ve}$$

$g''(x) > 0$

$g'(x)\uparrow$ing $\Rightarrow$ $g'(x) > g'(1)$

$g'(x) > 0$ $\Rightarrow$ $g(x)$ is increasing

$g(x) > g(1)$ $\Rightarrow$ $g(x) > 0$

H.P.

Sol.37 $f(x) = \tan^2 x + 6 \ln \sec x + 2 \cos x + 4 - 6 \sec x$

do upto $f'(x)$

Sol.38 Let $f(x) = \ell n\,(1+x) - \dfrac{x}{1+x}$ $D_f : x \in (-1, \infty)$

$$f'(x) = \frac{1}{1+x} - \frac{(1+x)-x}{(1+x)^2} = \frac{x}{(1+x)^2}$$

$f(x) \geq f(0)$ $\quad\forall\, x \in (-1, \infty)$

$f(x) \geq 0$ $\quad\forall\, x \in (-1, \infty)$

$f(x) > 0$ $\quad\forall\, x \in (-1, 0) \cup (0, \infty)$

Sol.39 $(a-x)^3 + (b-x)^3$ $\quad x < a$

$(x-a)^3 + (x-b)^3$ $\quad x \geq b$

$(x-a)^3 - (x-b)^3$ $\quad a \leq x < b$

$f'(x) = 3\,(x-a)^2 - 3\,(x-b)^2 = 0$

$(x-a)^2 = (x+b)^2$

$x - a = x - b$ non sense

$x - a = b - x$ $\quad \Rightarrow x = \dfrac{a+b}{2}$

$$f\left(\frac{a+b}{2}\right) = \left(\frac{a+b}{2} - a\right)^3 - \left(\frac{a+b}{2} - b\right)^3$$

$$= \frac{(b-a)^3}{8} - \frac{(a-b)^3}{8}$$

$$= \frac{2(b-a)^3}{8} = \frac{(b-a)^3}{4}$$